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On the Generalized Spectra of Topological Algebras

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1. INTRODUCTION

Local equicontinuity of the generalized spectra of certain topological algebras implies that the spectra in question coincide as topological spaces with the generalized spectra of the respective completed topological algebras [7]. On the other hand, for a fairly large class of topological algebras (i.e., m-barreled locally convex algebras), local equicontinuity of the (usual) spectrum is equivalent to its local compactness [6].

In this paper we are concerned with generalized spectra of (nonnecessarily commutative [11]) topological algebras. The results obtained have a special bearing, for the case under consideration, on the corresponding results contained in [6, 7, 11], which were also the motivation to the present setting. Thus, let E and F be topological algebras and let $\mathscr{M}(E, F)$ be the generalized spectrum of E (for F given) (i.e., the set of nonsero continuous algebra homomorphisms of E into F topologized as a subset of $\mathscr{L}_s(E, F)$; cf. Section 3 below). It is shown that if E is an m-barreled locally convex algebra and F a locally m-convex semiMontel one, local equicontinuity of $\mathscr{M}(E, F)$ is equivalent to its local compactness (cf., Theorem 3.2). Furthermore, we are dealing with representations of the generalized spectra of topological tensor product algebras. In this respect, it is proved that there exists a bicontinuous injection of the generalized spectrum of a locally convex infinite tensor product algebra into the cartesian product space of the spectra of the factor algebras (cf., Theorem 5.2). Moreover, by considering completed topological algebras, we still assume local equicontinuity for the generalized spectra. This assumption yields analogous representations of the generalized spectra of certain completed locally convex infinite tensor product algebras (with continuous multiplication) (Theorem 5.3).

On the other hand, with the class of m-barreled locally convex algebras several aspects concerning the continuity of the Gel'fand map are closely related [6]. Thus, we are also interested therein in the continuity of the generalized Gel'fand map (cf. Section 4). The results obtained constitute extended forms of those of [6; Section 3].

2. Preliminaries on Topological Algebras

All vector spaces and (associative linear) algebras considered in the following are over the field of complex numbers. The topological spaces involved are assumed to be Hausdorff unless otherwise indicated.

By a topological (resp. locally convex) algebra we mean an algebra E, equipped with a (Hausdorff) topology such that the (underlying) vector space E is a topological (resp. locally convex) vector space and the multiplication in E is separately continuous (considered as a bilinear map of $E \times E$ into E). A particular case of locally convex algebras, used in the following, are the locally m-convex ones [8]. Such an algebra has a (jointly) continuous mutliplication [8]. Now, by an m-barreled algebra we mean a (locally convex) algebra E such that every m-barrel (i.e., absorbing, balanced, convex, idempotent and closed subset of E) is a neighborhood of 0 in E [6]. For a detailed discussion concerning m-barreled algebras we also refer to [6].

A subset B of a locally convex algebra E is called m-bornivorous if it is idempotent and absorbs every bounded subset of E. Now, E is said to be m-infrabarreled if every m-bornivorous m-barrel of it is a neighborhood of 0 in E [6].

Finally, by a semiMontel algebra we mean a (locally convex) algebra E such that the locally convex space E is a semiMontel space [4].

3. The Generalized Spectrum of a Topological Algebra

Let E and F be topological algebras and let $\mathscr{L}_s(E, F)$ be the space of continuous linear maps between the topological vector spaces E and F, equipped with the topology of simple convergence in E. By the generalized spectrum of E (for F given) we mean the set $\mathscr{M}(E, F)$ of nonzero continuous (algebra) homomorphisms of E into F, equipped with the topology induced on it by $\mathscr{L}_s(E, F)$ [7]. If E and F have identity elements, then the elements of $\mathscr{M}(E, F)$ are assumed to be *identity preserving*.

We now prove the following.

Theorem 3.1 (Alaoglu–Bourbaki). *Let E, F be locally convex spaces such that F be a semiMontel space* [4]. *Then, every equicontinuous subset H of $\mathscr{L}_s(E, F)$ is relatively compact.*

Proof. First, H is (simply) bounded [9; p. 83, Corollary 4.1]. Thus, for each $x \in E$, $H(x) = \bigcup_{u \in H} u(x)$ is, by hypothesis for F, relatively compact in F, so that the assertion is now obtained by [1; p. 23, Corollary of Proposition 4].

PROPOSITION 3.1. *Let E be an m-barreled algebra and let F be a locally m-convex algebra. Then, every (simply) bounded subset B of the corresponding generalized spectrum $\mathcal{M}(E,F)$ is equicontinuous.*

Proof. Let W be a balanced, convex, idempotent and closed neighborhood of 0 in F. Then, $T = \bigcap_{u \in B} u^{-1}(W)$ is a barrel in E [9; p. 83, Theorem 4.3]. On the other hand, by $B \subseteq \mathcal{M}(E,F)$ and the fact that W is idempotent, T is an idempotent subset of E, that is an m-barrel, which yields the assertion, by hypothesis for E.

Let E and F be topological algebras and let $\mathcal{M}(E,F)$ be the generalized spectrum of E (for F given). A subset H of $\mathcal{M}(E,F)$ is said to be *locally equicontinuous* if for every element f of H there exists a neighborhood U of f in $\mathcal{M}(E,F)$ such that U is an equicontinuous subset of $\mathcal{M}(E,F)$ [7].

Now, the following theorem, which concerns the equivalence of local equicontinuity and local compactness of the generalized spectrum, constitutes an extended form of [6; p. 302, Theorem 2.1]. Thus, we have,

THEOREM 3.2. *Let E be an m-barreled algebra, let F be a locally m-convex algebra and let $\mathcal{M}(E,F)$ be the corresponding generalized spectrum of E. Consider the following statements:*

(1) *$\mathcal{M}(E,F)$ is a locally compact (Hausdorff) space.*

(2) *$\mathcal{M}(E,F)$ is locally equicontinuous.*

Then, (1) *implies* (2). *Moreover, if the algebras E and F have identity elements and F is a semiMontel algebra (Section* 2), *then the converse implication is also true.*

Proof. The assertion is an easy consequence of Theorem 3.1 and Proposition 3.1 above, by taking into account (concerning the proof of (2) implies (1)) the fact that $\mathcal{M}(E,F)$ is (simply) closed, so that we omit the details (cf. also [6; p. 302, Theorem 2.1] for the arguments).

4. CONTINUITY OF THE GENERALIZED GEL'FAND MAP

Let E and F be locally convex algebras and let $\mathcal{M}(E,F)$ be the corresponding generalized spectrum of E. The generalized Gel'fand transform of an element $x \in E$ is the map $\hat{x}: \mathcal{M}(E,F) \mapsto F: h \mapsto \hat{x}(h) = h(x)$. By the definition of the topology of $\mathcal{M}(E,F)$, it easily follows that each $\hat{x}$ is continuous. On the other hand, we consider the algebra $\mathscr{C}(\mathcal{M}(E,F),F)$ of continuous F-valued maps on $\mathcal{M}(E,F)$. Now, the *generalized Gel'fand map* is the (algebra) homomorphism g, defined by the relation

$$g: E \mapsto \mathscr{C}(\mathcal{M}(E,F),F): x \mapsto g(x) = \hat{x}. \tag{4.1}$$

Concerning the continuity of the generalized Gel'fand map g, one has the following theorem, which extends it [6; p. 305, Theorem 3.1].

THEOREM 4.1. *Let E and F be locally convex algebras. Then, the corresponding generalized Gel'fand map* $g: E \mapsto \mathscr{C}_c(\mathscr{M}(E, F), F)$, *the algebra* $\mathscr{C}(\mathscr{M}(E, F), F)$ *being equipped with the topology of compact convergence in* $\mathscr{M}(E, F)$, *is continuous if and only if every compact subset of* $\mathscr{M}(E, F)$ *is equicontinuous.*

Proof. The assertion is an easy consequence of [2; p. 23, Proposition 1], by applying the arguments given in [6; p. 306, Theorem 3.1], so that we omit the details.

On the other hand, as consequences of Proposition 3.1 and Theorem 4.1 above we now have the following results (cf. also [6; p. 306 Corollaries 3.1 and 3.2]).

COROLLARY 4.1. *Let E be an m-barreled algebra and let F be a locally m-convex algebra. Then, the corresponding generalized Gel'fand map g, defined by* (4.1) *above, is continuous.*

COROLLARY 4.2. *Let E be a quasicomplete m-infrabarreled locally convex algebra* (*Section* 2) *and let F be a locally m-convex algebra. Then, the respective generalized Gel'fand map g is continuous.*

Proof. Let H be a compact subset of $\mathscr{M}(E, F)$. Then, H being (simply) bounded, it is, by hypothesis for E, strongly bounded [9; p. 82, Corollary 3.4]. Consider, moreover, a balanced, convex, idempotent and closed neighborhood W of 0 in F. Then, $T = \bigcap_{u \in H} u^{-1}(W)$ is an m-barrel in E. Now, let B be a bounded subset of E. Then, the set $V(B, W)$ of $u \in \mathscr{L}(E, F)$ with $u(B) \subseteqq W$ is a neighborhood of 0 in $\mathscr{L}_b(E, F)$, the space $\mathscr{L}(E, F)$ being equipped with the topology of bounded convergence in E, and hence there exists $\lambda > 0$ with $H \subseteqq \lambda V(B, W)$, which obviously implies $\lambda B \subseteqq T$. Thus, by hypothesis for E, T is a neighborhood of 0 in E, hence H is equicontinuous, so that the assertion follows by Theorem 4.1, and the proof is finished.

5. THE GENERALIZED SPECTRUM OF A TOPOLOGICAL TENSOR PRODUCT ALGEBRA

In this section we shall use the terminology and some basic results of [5, 7, 10, 11]. Let E, F and G be topological algebras and let $f \in \mathscr{M}(E, G)$ and $g \in \mathscr{M}(F, G)$. Then, the map $f \times g: E \times F \mapsto G$, defined by

$$(x, y) \mapsto (f \times g)(x, y) = f(x) g(y), \tag{5.1}$$

is obviously a (separately continuous) bilinear map. Thus, there exists the corresponding tensor product (linear) map $f \otimes g$, which is also an (algebra) homomorphism in case of G commutative.

Now, let E and F be topological algebras. By a *compatible topology* on the corresponding tensor product algebra $E \otimes F$ we mean a (Hausdorff) topology τ on $E \otimes F$, making it a topological algebra, (notation $E \otimes_\tau F$), such that the canonical bilinear map of $E \times F$ into $E \otimes_\tau F$ is separately continuous. On the other hand, a (compatible) topology τ on $E \otimes F$ is called *enough* if the completion $E \widehat{\otimes}_\tau F$ of the (underlying) topological vector space $E \otimes_\tau F$ is a topological algebra. This is the case, however, for (compatible) topologies τ such that $E \otimes_\tau F$ has continuous multiplication.

We are interested in the sequel in compatible topologies τ such that the following conditions are also satisfied:

(5.a) For every locally convex algebra G and for any $f \in \mathscr{M}(E, G)$ and $g \in \mathscr{M}(F, G)$, $f \otimes g$ $\mathscr{L}_s(E \otimes_\tau F, G)$.

(5.b) For every locally convex algebra G and for any equicontinuous subsets $A \subseteqq \mathscr{M}(E, G)$ and $B \subseteqq \mathscr{M}(F, G)$, $A \otimes B$ is an equicontinuous subset of $\mathscr{L}_s(E \otimes_\tau F, G)$. (This obviously implies condition (5.a)).

If, in particular, E and F are locally convex algebras, the projective tensorial topology π on $E \otimes F$ is a compatible (enough) topology satisfying the conditions (5.a) and (5.b) [7, Section 2; 11, Section 3].

Now, let E, F be topological algebras with identity elements and let h be an element of $\mathscr{M}(E \otimes_\tau F, G)$, where τ is a compatible topology on $E \otimes F$ satisfying condition (5.a) and G a topological algebra. Then, there exists an element $(f, g)_h$ of $\mathscr{M}(E, G) \times \mathscr{M}(F, G)$, defined by the relations

$$f(x) = h(x \otimes 1), \quad x \in E \qquad \text{and} \qquad g(y) = h(1 \otimes y), \quad y \in F, \tag{5.2}$$

(with 1 standing for the identity elements of E, F respectively), so that the map,

$$u\colon \mathscr{M}\left(E \underset{\tau}{\otimes} F, G\right) \mapsto \mathscr{M}(E, G) \times \mathscr{M}(F, G)\colon h \mapsto u(h) = (f, g)_h\,, \tag{5.3}$$

may be defined.

Now, the following theorem is fundamental for the sequel (cf. [7, Theorems 2.1 and 3.1; 11, Theorems 3.1 and 3.3] for the arguments).

THEOREM 5.1 (A. Mallios [7]). *Let E, F and G be locally convex algebras all having identity elements and let G be with (jointly) continuous multiplication.*

Furthermore, let τ be a compatible topology on $E \otimes F$ satisfying condition (5.a). *Then, there exists a bicontinuous injection*

$$u\colon \mathcal{M}\left(E \underset{\tau}{\otimes} F, G\right) \mapsto \mathcal{M}(E, G) \times \mathcal{M}(F, G),$$

defined by (5.3), *whose range space is the subset Q of $\mathcal{M}(E, G) \times \mathcal{M}(F, G)$ consisting of all (f, g) such that $f(x)\,g(y) = g(y)\,f(x)$ in G for each $(x, y) \in E \times F$. Moreover, if E and F have continuous multiplication, τ is a* (*compatible*) *enough topology satisfying condition* (5.b) *and G is complete, then there exists a bicontinuous injection*

$$\hat{u}\colon \mathcal{M}\left(E \underset{\tau}{\hat{\otimes}} F, G\right) \mapsto \mathcal{M}(E, G) \times \mathcal{M}(F, G),$$

whose range is the set Q already defined. In particular, u (*resp.* $\hat{u}$) *is onto* (*homeomorphism*) *for G commutative.*

Furthermore, let $(E_\alpha, f_{\beta\alpha})$ be an inductive system (with respect to I) of locally convex algebras [5] and let $E = \varinjlim(E_\alpha, f_{\beta\alpha})$ (or briefly $E = \varinjlim E_\alpha$) be the corresponding inductive limit locally convex algebra [5]. If G is a topological algebra, then by standard arguments given, for example, in [10; p. 16, Proposition 4.1], one has

$$\mathcal{L}_s(E, G) = \varprojlim \mathcal{L}_s(E_\alpha, G) \tag{5.4}$$

within a topological isomorphism of the topological spaces involved.

In particular, consider the generalized spectra $\mathcal{M}(E, G)$ and $\mathcal{M}(E_\alpha, G)$, $\alpha \in I$, and suppose that the following conditions are satisfied:

For every $\alpha, \beta \in I$ with $\alpha \leqslant \beta$ and $h \in \mathcal{M}(E_\beta, G)$, (5.5)

$$\mathrm{Im}(f_{\beta\alpha}) \cap \complement\, \mathrm{Ker}(h) \neq \varnothing.$$

For every $\alpha \in I$ and $h \in \mathcal{M}(E, G)$, (5.6)

$$\mathrm{Im}(f_\alpha) \cap \complement\, \mathrm{Ker}(h) \neq \varnothing.$$

Then, by (5.4), one obtains

$$\mathcal{M}(E, G) = \varprojlim \mathcal{M}(E_\alpha, G) \tag{5.7}$$

within a homeomorphism (cf., also [11, (4.3)]).

We remark that the preceding conditions (5.5) and (5.6) are also necessary for (5.7) to hold. These conditions are fulfilled if the algebras considered have identity elements and the (algebra) homomorphisms are identity preserving. In particular, this meets the situation one has in the case of an infinite tensor product algebra [5].

Now, the next proposition is needed for what follows.

PROPOSITION 5.1. *Let E be a locally convex algebra, inductive limit of an inductive system $(E_\alpha, f_{\beta\alpha})$ (with respect to I) of locally convex algebras with continuous multiplication. Then, E has also continuous multiplication.*

Proof. If $\alpha, \beta \in I$, then there exists $\gamma \in I$ with $\gamma \geqslant \alpha$, $\gamma \geqslant \beta$ such that $f_\alpha \times f_\beta = f_\gamma \circ (f_{\gamma\alpha} \times f_{\gamma\beta})$ (cf. also (5.1) above), so that the assertion follows by hypothesis and [3; p. 490, Section 7.7.11].

Now, let $(E_i)_{i\in I}$ be a family of locally convex algebras with identity elements. In the sequel, we denote by J the set of finite subsets of I. Let $\alpha \in J$ and σ be an element of $\mathrm{Aut}(\alpha)$. Then, for every element (f_i) of $\prod_{i\in\alpha} \mathscr{M}(E_i, G)$, where G is a locally convex algebra, we consider the map

$$x^\sigma f_i\colon \prod_{i\in\alpha} E_i \mapsto G\colon (x_i) \mapsto x^\sigma f_i((x_i)) = \bigtimes_{i\in\alpha} f_{\sigma(i)}((x_{\sigma(i)})), \tag{5.8}$$

($\bigtimes_{i\in\alpha}$ denotes multiplication in G). Furthermore, we denote by Q_α the subset of $\prod_{i\in\alpha} \mathscr{M}(E_i, G)$ consisting of all elements (f_i) such that the corresponding maps $x^\sigma f_i$, defined by (5.8), coincide for all $\sigma \in \mathrm{Aut}(\alpha)$. If the algebras E_i, $i \in I$ have continuous multiplication, then each algebra $E_\alpha = \bigotimes_{i\in\alpha} E_i$, $\alpha \in J$ [5] has also continuous multiplication, so that the completion $\hat{E}_\alpha$ is a locally convex algebra. Thus, as it is easily verified, Theorem 5.1 can be extended to the case of algebras E_α, $\alpha \in J$. That is, we have

$$\mathscr{M}(E_\alpha, G) = Q_\alpha \quad \text{and} \quad \mathscr{M}(\hat{E}_\alpha, G) = Q_\alpha \tag{5.9}$$

within a homeomorphism respectively.

We are now in a position to state the main result of this section, which constitutes an extended form of Theorem 5.1 above to the case of topological infinite tensor product algebras.

THEOREM 5.2. *Let $(E_i)_{i\in I}$ be a family of locally convex algebras with identity elements, let $E = \bigotimes_{i\in I} E_i$ be the corresponding locally convex (projective) infinite tensor product algebra* [5] *and let G be a locally convex algebra with an identity element and continuous multiplication. Then, there exists a bicontinuous injection*

$$u\colon \mathscr{M}\Big(\bigotimes_{i\in I} E_i, G\Big) \mapsto \prod_{i\in I} \mathscr{M}(E_i, G),$$

whose range is the space $\varprojlim Q_\alpha$, such that u is onto (homeomorphism) for G commutative.

Proof. By the definition of E [5; p. 217, Definition 4.1], one has

$$E = \varinjlim E_\alpha = \varinjlim \left(\bigotimes_{i \in \alpha} E_i \right), \qquad \alpha \in J.$$

Hence, by (5.7) above, $\mathscr{M}(E, G) = \varprojlim \mathscr{M}(E_\alpha, G)$ (within a homeomorphism). On the other hand, by (5.9), $\mathscr{M}(E_\alpha, G) = Q_\alpha$ (within a homeomorphism), so that (cf. also [5; p. 218, footnote 1]) we finally have

$$\mathscr{M}(E, G) = \varprojlim Q_\alpha \subseteq \prod_{i \in I} \mathscr{M}(E_i, G),$$

which proves the assertion.

By using the arguments of [11; Proposition 4.1] one obtains the following.

Proposition 5.2. *Let $(E_i)_{i \in I}$ be a family of locally convex algebras all having identity elements and let E be the corresponding locally convex (projective) infinite tensor product algebra* [5]. *Suppose that each of the algebras E_i, $i \in I$ has a locally equicontinuous generalized spectrum $\mathscr{M}(E_i, G)$, where G is a locally convex algebra satisfying the conditions of Theorem* 5.2 *above, such that for all except of a finite many indices $i \in I$, the corresponding algebras have an equicontinuous generalized spectrum. Then, the generalized spectrum $\mathscr{M}(E, G)$ of the algebra E is locally equicontinuous.*

On the other hand, we have.

Theorem 5.3. *Under the assumptions of Theorem* 5.2 *above, suppose that each algebra E_i, $i \in I$ has continuous multiplication and a locally equicontinuous generalized spectrum such that for all but finite many i's, the corresponding algebras have an equicontinuous generalized spectrum. Moreover, let G be complete. Then, there exists a bicontinuous injection*

$$\hat{u}\colon \mathscr{M}\left(\hat{\bigotimes_{i \in I}} E_i, G \right) \mapsto \prod_{i \in I} \mathscr{M}(E_i, G),$$

whose range is the space $\varprojlim Q_\alpha$, considered in the statements of Theorem 5.2 *above, such that $\hat{u}$ is onto for G commutative.*

Proof. By Propositions 5.1 and 5.2 and Theorem 5.1, the assertion is easily reduced to that of Theorem 5.2, so that we omit the details.

Finally, we obtain the following

Theorem 5.4. *Let $(E_i)_{i \in I}$ be a family of m-barreled algebras with identity elements and let E be the corresponding locally convex (projective) infinite tensor product algebra* [5]. *Furthermore, let G be a commutative locally m-convex* *semiMontel algebra (Section 2) with an identity element. Then,* $\mathcal{M}(E, G)$ *is locally equicontinuous if and only if all* $\mathcal{M}(E_i, G)$, $i \in I$, *are equicontinuous except of a finite many of them, which are locally equicontinuous. In particular,* $\mathcal{M}(E, G)$ *is equicontinuous if and only if this is the case for each* $\mathcal{M}(E_i, G)$, $i \in I$.

Proof. Suppose that $\mathcal{M}(E, G)$ is locally equicontinuous. Then, by Theorem 3.2, $\mathcal{M}(E, G)$ is locally compact, so that the assertion follows by Theorem 5.2 and Tychonov's Theorem. On the other hand, the converse follows easily by Proposition 5.2. The rest part of the assertion is now obtained by standard arguments given in [5; p. 216, Proposition 3.2], so that we omit the details.

References

1. N. Bourbaki, "Espaces vectorielles topologiques," Chap. III-V, Hermann, Paris, 1964.
2. N. Bourbaki, "Topologie Générale," Chap. 10, Hermann, Paris, 1961.
3. R. E. Edwards, "Functional Analysis (Theory and Applications)," Holt, Rinerhart & Winston, New York, 1965.
4. J. Horvath, "Topological Vector Spaces and Distributions, I," Addison–Wesley, Reading, Massachusetts, 1966.
5. A. Mallios, Inductive limits and tensor products of topological algebras, *Math. Ann.* **170** (1967), 214–220.
6. A. Mallios, On the spectra of topological algebras, *J. Functional Analysis* **3** (1969), 301–309.
7. A. Mallios, On generalized spectra of topological tensor algebras, *Prakt. Akad. Athēnōn* **45** (1970), 76–82.
8. E. A. Michael, Locally multiplicatively-convex topological algebras, *Mem. Amer. Math. Soc.* **24** (1967), 1–83.
9. H. H. Schaefer, "Topological Vector Spaces," MacMillan, New York, 1966.
10. L. Tsitsas, Inductive limits and tensor products of ordered topological vector spaces and algebras, *Bull. Soc. Math. Grèce* **11** (1970), 6–30.
11. L. Tsitsas, On the generalized positive spectrum of ordered topological tensor product algebras, *J. Math. Anal. Appl.* **37** (1972), 341–350.